Derivation of a Blended BV

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Lets start with a MBV on animal i, m_i, and its breeding value, u_i. The first two moments $u_i, m_i, \hat{u}_i$ are

$$\begin{pmatrix} u_i \\ m_i \\ \hat{u}_i \end{pmatrix} \sim \left(\begin{pmatrix} 0 \\ M \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_u^2 & \sigma_{um} & \sigma_u^2 - PEV_i \\ \sigma_{um} & \sigma_m^2 & \frac{\sigma_{um}}{\sigma_u^2}(\sigma_u^2 - PEV_i) \\ \sigma_u^2 - PEV_i & \frac{\sigma_{um}}{\sigma_u^2}(\sigma_u^2 - PEV_i) & \sigma_u^2 - PEV_i \end{pmatrix} \right).$$

The BLUP of u_i given m_i and $\hat{u}_i$ is

$$\begin{aligned}
\tilde{u}_i &= \begin{pmatrix} \sigma_{um} & \sigma_u^2 - PEV_i \end{pmatrix} \begin{pmatrix} \sigma_m^2 & \frac{\sigma_{um}}{\sigma_u^2}(\sigma_u^2 - PEV_i) \\ \frac{\sigma_{um}}{\sigma_u^2}(\sigma_u^2 - PEV_i) & \sigma_u^2 - PEV_i \end{pmatrix}^{-1} \begin{pmatrix} m_i - M \\ \hat{u}_i \end{pmatrix} \\
&= \frac{\begin{pmatrix} \sigma_{um} & \sigma_u^2 - PEV_i \end{pmatrix}}{\sigma_m^2(\sigma_u^2 - PEV_i) - \frac{\sigma_{um}^2}{\sigma_u^4}[\sigma_u^2 - PEV_i]^2} \begin{pmatrix} \sigma_u^2 - PEV_i & -\frac{\sigma_{um}}{\sigma_u^2}(\sigma_u^2 - PEV_i) \\ -\frac{\sigma_{um}}{\sigma_u^2}(\sigma_u^2 - PEV_i) & \sigma_m^2 \end{pmatrix} \begin{pmatrix} m_i - M \\ \hat{u}_i \end{pmatrix} \\
&= (\sigma_u^2 - PEV_i) \frac{\begin{pmatrix} \sigma_{um} - \frac{\sigma_{um}}{\sigma_u^2}(\sigma_u^2 - PEV_i) & \sigma_m^2 - \frac{\sigma_{um}^2}{\sigma_u^2} \end{pmatrix}}{\sigma_m^2(\sigma_u^2 - PEV_i) - \frac{\sigma_{um}^2}{\sigma_u^4}[\sigma_u^2 - PEV_i]^2} \begin{pmatrix} m_i - M \\ \hat{u}_i \end{pmatrix} \\
&= \frac{\begin{pmatrix} \sigma_{um} - \frac{\sigma_{um}}{\sigma_u^2}(\sigma_u^2 - PEV_i) & \sigma_m^2 - \frac{\sigma_{um}^2}{\sigma_u^2} \end{pmatrix}}{\sigma_m^2 - \frac{\sigma_{um}^2}{\sigma_u^4}[\sigma_u^2 - PEV_i]} \begin{pmatrix} m_i - M \\ \hat{u}_i \end{pmatrix} \\
&= \frac{\begin{pmatrix} \frac{\sigma_{um}}{\sigma_u^2} PEV_i & \sigma_r^2 \end{pmatrix}}{\sigma_r^2 + \frac{\sigma_{um}^2}{\sigma_u^4} PEV_i} \begin{pmatrix} m_i - M \\ \hat{u}_i \end{pmatrix} \\
&= \frac{\frac{\sigma_{um}}{\sigma_u^2} PEV_i}{\sigma_r^2 + \frac{\sigma_{um}^2}{\sigma_u^4} PEV_i}(m_i - M) + \frac{\sigma_r^2}{\sigma_r^2 + \frac{\sigma_{um}^2}{\sigma_u^4} PEV_i} \hat{u}_i
\end{aligned}$$

Replace PEV/σ_u^2 with $1 - R_i^2$ where R_i^2 is the reliability of the BV for animal i,

$$\begin{aligned}
\tilde{u}_i &= \frac{(1 - R_i^2)}{(1 - r_g^2) + r_g^2(1 - R_i^2)} \left[\frac{\sigma_{um}}{\sigma_m^2}(m_i - M) \right] + \frac{1 - r_g^2}{(1 - r_g^2) + r_g^2(1 - R_i^2)} \hat{u}_i \\
&= \frac{(1 - R_i^2)}{(1 - r_g^2) + r_g^2(1 - R_i^2)} \hat{m}_i + \frac{1 - r_g^2}{(1 - r_g^2) + r_g^2(1 - R_i^2)} \hat{u}_i \\
&= \frac{1 - R_i^2}{(1 - r_g^2 R_i^2)} \hat{m}_i + \frac{1 - r_g^2}{(1 - r_g^2 R_i^2)} \hat{u}_i
\end{aligned}$$

where $\hat{m}_i = \frac{\sigma_{um}}{\sigma_m^2}(m_i - M)$ is the BLUP of u_i given the MBV.

The above assumes that u_i and $\hat{u}_i$ are both set at a base of zero. This can be generalized to having a base B for u_i and $\hat{u}_i$,

$$\begin{aligned}\tilde{u}_i &= B + \frac{1-R_i^2}{(1-r_g^2R_i^2)}\hat{m}_i + \frac{1-r_g^2}{(1-r_g^2R_i^2)}(\hat{u}_i - B),\\ &= \hat{u}_i + \frac{1-R_i^2}{(1-r_g^2R_i^2)}[\hat{m}_i - r_g^2(\hat{u}_i - B)].\end{aligned}$$

The variance of $\tilde{u}_i$ is

$$\begin{aligned}\text{Var}(\tilde{u}_i) &= \text{Var}\left(\frac{1-R_i^2}{(1-r_g^2R_i^2)}\hat{m}_i + \frac{1-r_g^2}{(1-r_g^2R_i^2)}\hat{u}_i\right)\\ &= \frac{(1-R_i^2)^2r_g^2 + (1-r_g^2)^2R_i^2 + 2(1-R_i^2)(1-r_g^2)r_g^2R_i^2}{(1-r_g^2R_i^2)^2}\sigma_u^2\\ &= \frac{(1-R_i^2)^2r_g^2 + (1-R_i^2)(1-r_g^2)r_g^2R_i^2 + (1-r_g^2)^2R_i^2 + (1-R_i^2)(1-r_g^2)r_g^2R_i^2}{(1-r_g^2R_i^2)^2}\sigma_u^2\\ &= \frac{(1-R_i^2)r_g^2[1-r_g^2R_i^2] + (1-r_g^2)R_i^2[1-r_g^2R_i^2]}{(1-r_g^2R_i^2)^2}\sigma_u^2\\ &= \frac{(1-R_i^2)r_g^2 + (1-r_g^2)R_i^2}{(1-r_g^2R_i^2)}\sigma_u^2\end{aligned}$$

The reliability of the $\tilde{u}_i$ is

$$\begin{aligned}\text{Rel}(\tilde{u}_i) &= \left[\frac{(1-R_i^2)r_g^2 + (1-r_g^2)R_i^2}{(1-r_g^2R_i^2)}\right]\\ &= R_i^2 + r_g^2(1-R_i^2)\left[\frac{1-R_i^2}{1-R_i^2r_g^2}\right].\end{aligned}$$

The BIF accuracy is

$$\begin{aligned}Acc(\tilde{u}_i) &= 1 - \sqrt{1 - \frac{(1-R_i^2)r_g^2 + (1-r_g^2)R_i^2}{(1-r_g^2R_i^2)}}\\ &= 1 - \sqrt{\frac{1-r_g^2-R_i^2+r_g^2R_i^2}{1-r_g^2R_i^2}}.\end{aligned}$$

Estimation of MBV base (M)

While the base adjustment to a breeding values is known, the base for the MBV needs to be estimated. One approach is to estimate it as a fixed effect in a correlated trait analysis. Alternatively, assuming independence the base can be estimated from a sample of animals which have both an EBV and an MBV. Letting $b = \sigma_{um}/\sigma_m^2$, the BLUE of M is

$$\begin{aligned}
\sum_i \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} r_g^2/b^2 & r_g^2 R_i^2/b \\ r_g^2 R_i^2/b & R_i^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \hat{M} &= \sum_i \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} r_g^2/b^2 & r_g^2 R_i^2/b \\ r_g^2 R_i^2/b & R_i^2 \end{pmatrix}^{-1} \begin{pmatrix} m_i \\ \hat{u}_i - B \end{pmatrix} \\
\sum_i \frac{b^2 R_i^2}{r_g^2 R_i^2 - r_g^4 R_i^4} \hat{M} &= \sum_i \frac{b^2}{r_g^2 R_i^2 - r_g^4 R_i^4} \begin{pmatrix} R_i^2 & -r_g^2 R_i^2/b \end{pmatrix} \begin{pmatrix} m_i \\ \hat{u}_i - B \end{pmatrix} \\
\sum_i \frac{1}{1 - r_g^2 R_i^2} \hat{M} &= \sum_i \frac{m_i - r_g^2(\hat{u}_i - B)/b}{1 - r_g^2 R_i^2} \\
\hat{M} &= \frac{\sum_i w_i [m_i - r_g^2(\hat{u}_i - B)/b]}{\sum_i w_i},
\end{aligned}$$

where

$$w_i = \frac{1}{1 - r_g^2 R_i^2}.$$

Accounting for Inflation due to training

The previous derivations assumed that the MBV on animal i and the estimated breeding value are conditionally independent given the breeding value. In practice, the same phenotypic information is used to both train the MBV and predict the animal's breeding value. The use of the same phenotypic data can result in the loss of conditional independence. The loss of independence can be modeled by introducing an inflation factor Δ into the covariance matrix. The first two moments $u_i, \hat{m}_i, \hat{u}_i$ can now be written as

$$\begin{pmatrix} u_i \\ \hat{m}_i \\ \hat{u}_i \end{pmatrix} \sim \left(\begin{pmatrix} B \\ 0 \\ B \end{pmatrix}, \sigma_u^2 \begin{pmatrix} 1 & r_g^2 & R_i^2 \\ r_g^2 & r_g^2 & r_g^2 R_i^2 \Delta \\ R_i^2 & r_g^2 R_i^2 \Delta & R_i^2 \end{pmatrix} \right).$$

The BLUP of u_i given $\hat{m}_i$ and $\hat{u}_i$ is

$$\begin{aligned}
\tilde{u}_i &= B + \begin{pmatrix} r_g^2 & R_i^2 \end{pmatrix} \begin{pmatrix} r_g^2 & r_g^2 R_i^2 \Delta \\ r_g^2 R_i^2 \Delta & R_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{m}_i \\ \hat{u}_i - B \end{pmatrix} \\
&= B + \frac{1}{r_g^2 R_i^2 - r_g^4 R_i^4 \Delta^2} \begin{pmatrix} r_g^2 & R_i^2 \end{pmatrix} \begin{pmatrix} R_i^2 & -r_g^2 R_i^2 \Delta \\ -r_g^2 R_i^2 \Delta & r_g^2 \end{pmatrix}^{-1} \begin{pmatrix} \hat{m}_i \\ \hat{u}_i - B \end{pmatrix} \\
&= B + \frac{1}{r_g^2 R_i^2 - r_g^4 R_i^4 \Delta^2} \begin{pmatrix} r_g^2 R_i^2 - r_g^2 R_i^4 \Delta & r_g^2 R_i^2 - r_g^4 R_i^2 \Delta \end{pmatrix} \begin{pmatrix} \hat{m}_i \\ \hat{u}_i - B \end{pmatrix} \\
&= B + \frac{1}{1 - r_g^2 R_i^2 \Delta^2} \begin{pmatrix} 1 - R_i^2 \Delta & 1 - r_g^2 \Delta \end{pmatrix} \begin{pmatrix} \hat{m}_i \\ \hat{u}_i - B \end{pmatrix} \\
&= B + \frac{1 - R_i^2 \Delta}{1 - r_g^2 R_i^2 \Delta^2} (\hat{m}_i) + \frac{1 - r_i^2 \Delta}{1 - r_g^2 R_i^2 \Delta^2} (\hat{u}_i - B) \\
&= \hat{u}_i + \frac{1 - R_i^2 \Delta}{1 - r_g^2 R_i^2 \Delta^2} \left[(\hat{m}_i) - r_g^2 \Delta (\hat{u}_i - B) \right].
\end{aligned}$$

The variance of $\tilde{u}_i$ is

$$
\begin{aligned}
\mathrm{Var}(\tilde{u}_i) &= \mathrm{Var}\left(\frac{1-R_i^2\Delta}{(1-r_g^2R_i^2\Delta^2)}\hat{m}_i + \frac{1-r_g^2\Delta}{(1-r_g^2R_i^2\Delta^2)}\hat{u}_i\right) \\
&= \frac{(1-R_i^2\Delta)^2r_g^2 + (1-r_g^2\Delta)^2R_i^2 + 2(1-R_i^2\Delta)(1-r_g^2\Delta)r_g^2R_i^2\Delta}{(1-r_g^2R_i^2\Delta^2)^2}\sigma_u^2 \\
&= \frac{(1-R_i^2\Delta)^2r_g^2 + (1-R_i^2\Delta)(1-r_g^2\Delta)r_g^2R_i^2 + (1-r_g^2\Delta)^2R_i^2 + (1-R_i^2\Delta)(1-r_g^2\Delta)r_g^2R_i^2}{(1-r_g^2R_i^2\Delta^2)^2}\sigma_u^2 \\
&= \frac{(1-R_i^2\Delta)r_g^2[1-r_g^2R_i^2\Delta] + (1-r_g^2\Delta)R_i^2[1-r_g^2R_i^2\Delta^2]}{(1-r_g^2R_i^2\Delta^2)^2}\sigma_u^2 \\
&= \frac{(1-R_i^2\Delta)r_g^2 + (1-r_g^2\Delta)R_i^2}{(1-r_g^2R_i^2\Delta^2)}\sigma_u^2.
\end{aligned}
$$

The reliability of the $\tilde{u}_i$ is

$$
\begin{aligned}
\mathrm{Rel}(\tilde{u}_i) &= \left[\frac{(1-R_i^2\Delta)r_g^2 + (1-r_g^2\Delta)R_i^2}{(1-r_g^2R_i^2\Delta^2)}\right] \\
&= R_i^2 + r_g^2(1-R_i^2\Delta)\left[\frac{1-R_i^2\Delta}{1-R_i^2r_g^2\Delta^2}\right].
\end{aligned}
$$

Derivation of the covariance between $\hat{m}_i$ and $\hat{u}_i$

Starting with the covariance matrix of u_i and $\hat{u}_i$,

$$
\mathrm{Cov}\begin{pmatrix} u_i \\ \hat{u}_i \end{pmatrix} = \begin{pmatrix} 1 & R_i^2 \\ R_i^2 & R_i^2 \end{pmatrix}\sigma_u^2.
$$

We can then write $\hat{u}_i$ as the BLUP of $\hat{u}_i|u_i$ and an uncorrelated residual

$$
\begin{aligned}
\hat{u}_i &= \frac{\mathrm{Cov}(\hat{u}_i, u_i)}{\mathrm{Var}(u_i)}u_i + \left[\hat{u}_i - \frac{\mathrm{Cov}(\hat{u}_i, u_i)}{\mathrm{Var}(u_i)}u_i\right] \\
&= R_i^2u_i + \left[\hat{u}_i - R_i^2u_i\right].
\end{aligned}
$$

The resulting covariance matrix is

$$
\mathrm{Cov}\begin{pmatrix} u_i \\ \hat{u}_i - R_i^2u_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & R_i^2(1-R_i^2) \end{pmatrix}\sigma_u^2.
$$

Similarly for $\hat{m}_i$ we get

$$
\mathrm{Cov}\begin{pmatrix} u_i \\ \hat{m}_i - r_g^2u_i \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r_g^2(1-r_g^2) \end{pmatrix}\sigma_u^2.
$$

Assuming the prediction errors for $\hat{u}_i$ and $\hat{m}_i$ given u_i are independent we get

$$
\mathrm{Cov}\begin{pmatrix} u_i \\ \hat{u}_i - R_i^2u_i \\ \hat{m}_i - r_g^2u_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R_i^2(1-R_i^2) & 0 \\ 0 & 0 & r_g^2(1-r_g^2) \end{pmatrix}\sigma_u^2.
$$

We can now write $\hat{u}_i$ and $\hat{m}_i$ in terms of u_i, $\hat{u}_i - R_i^2 u_i$, and $\hat{m}_i - r_g^2 u_i$

$$\begin{pmatrix} u_i \\ \hat{u}_i \\ \hat{m}_i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ R_i^2 & 1 & 0 \\ r_g^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_i \\ \hat{u}_i - R_i^2 u_i \\ \hat{m}_i - r_g^2 u_i \end{pmatrix}.$$

Which gives us

$$\begin{aligned}
\mathrm{Cov}\begin{pmatrix} u_i \\ \hat{u}_i \\ \hat{m}_i \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ R_i^2 & 1 & 0 \\ r_g^2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & R_i^2(1-R_i^2) & 0 \\ 0 & 0 & r_g^2(1-r_g^2) \end{pmatrix} \begin{pmatrix} 1 & R_i^2 & r_g^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sigma_u^2 \\
&= \begin{pmatrix} 1 & 0 & 0 \\ R_i^2 & R_i^2(1-R_i^2) & 0 \\ r_g^2 & 0 & r_g^2(1-r_g^2) \end{pmatrix} \begin{pmatrix} 1 & R_i^2 & r_g^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \sigma_u^2 \\
&= \begin{pmatrix} 1 & R_i^2 & r_g^2 \\ R_i^2 & R_i^2 & R_i^2 r_g^2 \\ r_g^2 & R_i^2 r_g^2 & r_g^2 \end{pmatrix} \sigma_u^2.
\end{aligned}$$